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2007 J. Phys. A: Math. Theor. 40 11255

(http://iopscience.iop.org/1751-8121/40/37/004)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **40** (2007) 11255–11269 doi:10.1088/1751-8113/40/37/004

Integer partitions and exclusion statistics

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Received 16 May 2007, in final form 31 July 2007
Published 29 August 2007
Online at stacks.iop.org/JPhysA/40/11255

Abstract
We provide a combinatorial description of exclusion statistics in terms of minimal difference p partitions. We compute the probability distribution of the number of parts in a random minimal p partition. It is shown that the bosonic point $p = 0$ is a repulsive fixed point for which the limiting distribution has a Gumbel form. For all positive p, the distribution is shown to be Gaussian.

PACS numbers: 05.30.Jp, 05.30.Pr, 02.10.Ox

1. Introduction

The integer partition problem has a long history going back to Euler. The classical question asks: In how many ways $\rho(E)$ can one partition an integer E into nonzero integer parts $E = \sum_j h_j$ such that $h_j \geqslant h_{j+1}$ for all $j = 1, 2, \ldots$? For example, $\rho(4) = 5$: $4 = 4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1$. Pictorially, one can represent the part h_j as the height of the j th column with nonincreasing heights such that the total height under the columns is E. Hardy and Ramanujan proved [1] that for large E, $\rho(E) \simeq \frac{1}{4}\frac{1}{3^{1/2}E}\,\mathrm{e}^{a\sqrt{E}}$, with $a = \pi\sqrt{2/3}$. Similarly, one can ask the number of ways of partitioning the integer E into *distinct* integer summands, i.e., $E = \sum_j h_j$ such that $h_j > h_{j+1}$ with strictly decreasing height. For example, the integer 4 can be partitioned into distinct summands in only two ways, $4 = 4 = 3 + 1$. In this restricted case, it is known that asymptotically for large E [2], $\rho(E) \simeq \frac{1}{4}\frac{1}{3^{1/4}E^{3/4}}\,\mathrm{e}^{b\sqrt{E}}$, where $b = a/\sqrt{2} = \pi/\sqrt{3}$.

Another way of representing integer partitions makes clear the connection with a gas of noninteracting quantum particles. Let n_i be the number of columns of height $h = i$ in a given partition, i.e., the number of times the summand i appears in a given partition. For example, in the partition $4 = 2 + 1 + 1$, one has $n_1 = 2, n_2 = 1$ and $n_j = 0$ for all $j > 2$. Then, $E = \sum_i n_i\epsilon_i$, where $\epsilon_i = i$ for $i = 1, 2, \ldots$ represents equidistant single particle energy levels and $n_i = 0, 1, 2, \ldots$ represents the occupation number of the ith level. In the unrestricted problem, the occupation number $n_i = 0, 1, 2\ldots$ (bosons) whereas in the restricted problem $n_i = 0, 1$ (fermions). Therefore, $E = \sum_i n_i \epsilon_i$ is the total energy of the system and

$$\rho(E) = \sum_{n_i} \delta\left(E - \sum_{i=1}^{\infty} n_i \epsilon_i\right). \tag{1}$$

If, in addition, one restricts the number of summands to be N, then the number $\rho(E, N)$ of ways of partitioning E into N parts is simply the micro-canonical partition function of a gas of quantum particles with total energy E and total number of particles N:

$$\rho(E, N) = \sum_{n_i} \delta\left(E - \sum_{i=1}^{\infty} n_i \epsilon_i\right) \delta\left(N - \sum_{i=1}^{\infty} n_i\right). \tag{2}$$

Evidently, $\rho(E) = \sum_{N=0}^{\infty} \rho(E, N)$. Even though the sum $\rho(E)$ has similar asymptotic behavior for large E for bosons and fermions, i.e., $\ln(\rho(E)) \sim \sqrt{E}$ (up to a constant prefactor), we will show in this paper that $\rho(E, N)$, as a function of N for a fixed E, has rather different behavior for bosons and fermions.

Thus, a gas of noninteracting bosons or fermions occupying a single particle equidistant spectrum ($\epsilon_i = i$) both have a combinatorial interpretation in terms of partitions of an integer E into N parts.

- Bose statistics corresponds to the case of unrestricted partitions $n_i = 0, 1, 2\ldots$.
- Fermi statistics corresponds to the case of restricted partitions with distinct summands $n_i = 0, 1$.

A natural question, that we address in this paper, is how to provide a combinatorial description of a quantum gas obeying exclusion statistics. Exclusion statistics is a generalization of Bose and Fermi statistics [4–8]. It has been found explicitly in quantum models of interacting particle systems, notably in the two-dimensional lowest-Landau-level (LLL) anyon model [5] (i.e., the anyon model projected into the LLL of a strong magnetic field) and the one-dimensional Calogero model [7, 9–13]. Note that the Calogero model can be obtained as a particular limit of the LLL anyon model [14], the latter being a particular exactly solvable projection of the anyon model: it follows that exclusion statistics is deeply rooted in the more general concept of anyon statistics [15]. Unlike the Bose and Fermi statistics which describes noninteracting particles, a combinatorial description of exclusion statistics is *a priori* quite nontrivial since the underlying physical models with exclusion statistics describe truly interacting N-body systems.

We show in this paper that a combinatorial interpretation of exclusion statistics involves a generalization of the partition problem known as the minimal difference partition (MDP) problem. In MDP, one partitions a positive integer E into N nonzero parts, $E = \sum_{j=1}^{N} h_j$ (with $h_j > 0$ for all $j = 1, 2, \ldots, N$) such that each summand exceeds the next by at least an integer p, i.e., $(h_j - h_{j+1}) \geqslant p$ for all $j = 1, 2, \ldots, N-1$. Therefore, $p = 0$ corresponds to unrestricted partitions (bosons) and $p = 1$ to restricted partitions (fermions) into distinct parts. Even though the parameter p in MDP is an integer, one can analytically continue the results to noninteger values of p and we will show that for $0 < p < 1$ the MDP corresponds to a gas of quantum particles obeying exclusion statistics.

Apart from establishing this equivalence between the MDP problem and exclusion statistics, we also provide a detailed analysis of the asymptotic behavior of $\rho_p(E, N)$, i.e., the number of ways the integer E can be partitioned into N parts in the MDP problem, for all $p \geqslant 0$. This analysis tells us how the variable N fluctuates from one partition to another for fixed E. Indeed, defining $\rho_p(E) = \sum_N \rho_p(E, N)$ as the total number of partitions of E

and treating all such partitions equally likely, the ratio $P_p(N|E) = \rho_p(E, N)/\rho_p(E)$ is the probability distribution of the random variable N, given E. We show that this distribution, properly centered and scaled, has rather different limiting shapes for $p = 0$ and $p > 0$. While for $p = 0$ the scaled distribution is asymmetric and has a Gumbel shape, for $p > 0$ (including the fermionic case $p = 1$) the scaled distribution is symmetric and has a Gaussian shape.

At this point, it may be useful to summarize our main mathematical results for the asymptotic behavior of $P_p(N|E)$. For the bosonic case ($p = 0$), the limiting shape of the distribution was first derived by Erdös and Lehner using rigorous methods involving upper and lower bounds [16]. In this paper, we calculate the limiting shapes of $P_p(N|E)$ for all $p \geqslant 0$. Moreover, our method allows us to compute the probabilities of *atypical large* fluctuations which go beyond the range of validity of the limiting distributions.

For $p = 0$ we show that $P_0(N|E)$, as a function of N for fixed E, has a peak at a characteristic value $N_0^*(E) \simeq \frac{1}{a}\sqrt{E}\log(4E/a^2)$ for large E, where $a = \pi\sqrt{2/3}$, and the random variable N *typically* fluctuates around $N_0^*(E)$ over a scale $\sim\sqrt{E}$. Moreover, in the vicinity of $N_0^*(E)$ over a range $|N - N_0^*(E)| \sim O(\sqrt{E})$, the distribution $P_0(N|E)$ has a scaling form (or a limiting law). In terms of the cumulative probability,

$$Q_0(N|E) = \sum_{N'=0}^{N} P_0(N'|E) \approx F_0\left(\frac{a}{2\sqrt{E}}(N - N_0^*(E))\right), \tag{3}$$

where the scaling function $F_0(z)$ has an asymmetric Gumbel form, thus recovering the Erdös–Lehner result [16]

$$F_0(z) = \exp[-\exp[-z]]. \tag{4}$$

In contrast, for $p > 0$, the distribution $P_p(N|E)$ has quite a different asymptotic behavior. It has a peak at a characteristic value $N_p^*(E) \simeq a_1(p)\sqrt{E}$ and N typically fluctuates around $N_p^*(E)$ over a scale of $\sim E^{1/4}$ for all p. Moreover, we show that, on this scale, the fluctuations are Gaussian. More precisely, we show that in the vicinity of $N_p^*(E)$ the cumulative probability $Q_p(N|E)$ has a scaling form

$$Q_p(N|E) \approx F\left(\frac{N - a_1(p)\sqrt{E}}{a_2(p)E^{1/4}}\right), \qquad \text{where} \quad F(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} \mathrm{e}^{-y^2/2}\,\mathrm{d}y \tag{5}$$

is a universal scaling function independent of p (>0). The two nonuniversal scale factors $a_1(p)$ and $a_2(p)$ however depend explicitly on p and can be computed exactly. For example, for fermions ($p = 1$), we recover the Erdös–Lehner result for the mean

$$a_1(1) = \frac{2\sqrt{3}}{\pi}\ln(2) \tag{6}$$

and get a new result for the variance

$$a_2(1) = \left[\frac{3\pi^2 - 36\ln^2(2)}{\sqrt{3}\pi^3}\right]^{1/2}. \tag{7}$$

Thus, as far as the limiting shape of the scaled distribution of $P_p(N|E)$ is concerned, it is a universal Gaussian for all $p > 0$. The fermionic case $p = 1$ is thus a representative of all $p > 0$ and can be considered as an attractive fixed point along the p-axis (see figure 1). In contrast, the bosonic case $p = 0$ represents a repulsive fixed point where the shape is Gumbel.

The limit laws above describe the probabilities of *typical* fluctuations of N around its characteristic value $N_p^*(E)$. In this paper, we have also investigated the probability of *atypical large* fluctuations of N away from $N_p^*(E)$ and calculated the corresponding large deviation

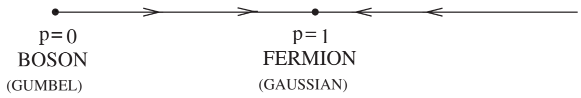

Figure 1. Schematic flows along the p-axis. The $p = 0$ represents the bosonic fixed point where the limiting distribution of $P_p(N|E)$ is Gumbel. In contrast, the behavior for all $p > 0$ is controlled by the fermionic fixed point at $p = 1$ where the limiting distribution is Gaussian.

functions exactly. Like the limit laws, the large deviation properties for $p > 0$ turns out to be rather different from the $p = 0$ case, thus confirming the fixed point picture of figure 1. Curiously though, we show that the large deviation function for any $p > 0$ is related to that of $p = 0$ via an exact nonlinear relation.

The paper is organized as follows. In section 2, we precisely define the MDP problem, provide an exact derivation of the generating function of $\rho_p(E, N)$ and establish a nonlinear relation between $\rho_p(E, N)$ with $p > 0$ and $\rho_0(E, N)$. In section 3, we show how the MDP problem with $0 < p < 1$ corresponds to exclusion statistics. In section 4, we provide a detailed asymptotic analysis of $\rho_p(E, N)$ for all $p \geqslant 0$ and obtain the limiting shapes of the scaled distribution $P_p(N|E)$ and also calculate exactly the associated large deviation functions. Finally, we conclude with a summary and open problems in section 5. An appendix follows where a grand-canonical approach to MDP partition asymptotics is presented.

2. Minimal difference partition problem: a combinatorial approach

In the MDP problem, one partitions an integer E into N nonzero parts, $E = \sum_{j=1}^{N} h_j$ (with $h_j > 0$ for each $j = 1, 2, \ldots, N$) such that each part exceeds the next one by at least an integer p, i.e., $(h_j - h_{j+1}) \geqslant p$ for all $j = 1, 2, \ldots, N - 1$ (see figure 2). Let $\rho_p(E, N)$ denote the number of ways one can achieve this. Clearly, the cases $p = 0$ and $p = 1$ reduce, respectively, to the unrestricted partitions (bosons) and the restricted partitions (fermions). The generating function for $\rho_p(E, N)$ is well known [3] and is given in equation (11). However, here we provide a simple derivation of this result that brings out in a direct way a nontrivial connection between the cases $p > 0$ and $p = 0$ which will be used later for the analysis of the asymptotic behavior.

Let us first establish an exact one-to-one correspondence between a partition configuration of the MDP with nonzero $p > 0$ and a partition configuration with $p = 0$. Let $\{h_j\}$ denote the set of nonzero heights in the partition of $E = \sum_{j=1}^{N} h_j$ for $p = 0$ (bosonic case). Thus, $h_j \geqslant h_{j+1}$ for all $j = 1, 2, \ldots, N - 1$. Let us now define a new set of heights $h'_j = h_j + p(N - j)$ for $j = 1, 2, \ldots, N$. Thus, $h'_j - h'_{j+1} = h_j - h_{j+1} + p$ for all $j = 1, 2, \ldots, N - 1$ and $h'_N = h_N > 0$. The new heights thus satisfy the constraint $(h'_j - h'_{j+1}) \geqslant p$ for all $j = 1, 2, \ldots, N - 1$ and their total height is given by

$$E' = \sum_{j=1}^{N} h'_j = E + pN(N-1)/2 = \sum_{j=1}^{N} h_j + pN(N-1)/2. \tag{8}$$

Therefore, the primed heights correspond to a partition configuration of the integer E' into N parts with $p > 0$. This exact correspondence then provides us with the following identity valid for all N:

$$\rho_p(E, N) = \rho_0\left(E - \frac{p}{2}N(N-1), N\right). \tag{9}$$

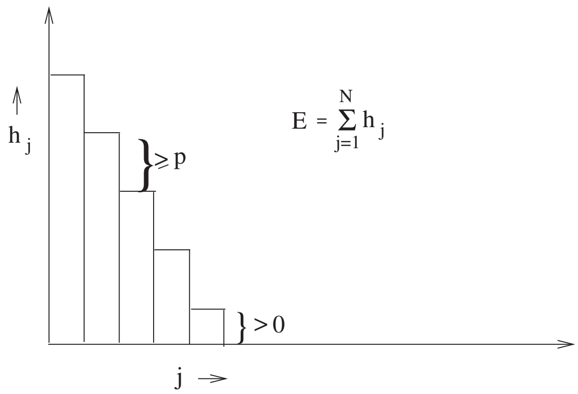

Figure 2. A typical partition configuration of the MDP problem with $N = 5$. The column heights $h_j > 0$ for all $j = 1, 2, \ldots, N$ and their total height is $E = \sum_{j=1}^{N} h_j$. In addition, they satisfy the constraint $(h_j - h_{j+1}) \geqslant p$ for an integer p for all $j = 1, 2, \ldots, N - 1$.

Thus, if one can compute the partition function $\rho_0(E, N)$ for the bosonic case ($p = 0$), this identity can be used to obtain exact results for any arbitrary $p > 0$, including the fermionic case $p = 1$.

For the bosonic ($p = 0$) case, a straightforward calculation gives the generating function

$$\sum_{E=1}^{\infty} \rho_0(E, N)x^E = \frac{x^N}{(1-x)(1-x^2)\cdots(1-x^N)}. \tag{10}$$

Using the correspondence in equation (9) gives the general result for all $p \geqslant 0$:

$$\sum_{E=1}^{\infty} \rho_p(E, N)x^E = \frac{x^{N+pN(N-1)/2}}{(1-x)(1-x^2)\cdots(1-x^N)}. \tag{11}$$

It turns out to be convenient sometimes to use the cumulative partition function $C_p(E, N) = \sum_{N'=0}^{N} \rho_p(E, N')$. For $p = 0$, its generating function can be easily derived from equation (10) and has a particularly simple form which turns out to be rather useful:

$$\sum_{E=1}^{\infty} C_0(E, N)x^E = \frac{1}{(1-x)(1-x^2)\cdots(1-x^N)}. \tag{12}$$

Comparing equations (11) and (12), one gets another identity

$$\rho_p(E, N) = C_0\left(E - N - \frac{p}{2}N(N-1), N\right) \tag{13}$$

which we will be using later.

3. The MDP with $0 < p < 1$ and exclusion statistics

In this section, we show that the MDP problem with integer parameter p, continued analytically to the range $0 \leqslant p \leqslant 1$, corresponds to a quantum gas of interacting particles obeying exclusion statistics. This correspondence is established at two levels: (i) at a microscopic level where we show in section 3.1 that $\rho_p(E, N)$ of the MDP problem corresponds precisely to the micro-canonical partition function of the one-dimensional Calogero model in an external harmonic potential and (ii) at a more general thermodynamical level in section 3.2.

3.1. Equivalence between the MDP problem and the spectrum of the Calogero model in a harmonic well

The aim of this subsection is to show that there is an exact one-to-one correspondence between the energy levels of the one-dimensional Calogero model in an external harmonic well and the partition configurations of the MDP problem, continued analytically to $0 \leqslant p \leqslant 1$ in the sense explained below.

The Calogero model (for a review, see [17, 18]) describes an interacting quantum particle system on a line where the particles attract each other by an inverse square potential. In order to have a proper thermodynamic limit, one can either put the N particles in a finite box of size L and then take the $N \to \infty$, $L \to \infty$ limit keeping the density N/L fixed. Alternatively, one can keep the particles on the infinite line, but switch on an external harmonic potential of strength ω. In the latter case, one has to eventually take the limit $\omega \to 0$ in a suitable way. It turns out that while the model in a box is not integrable, the model in a harmonic potential is integrable. Setting the Planck's constant $\hbar = 1$ and the mass of each particle $m = 1$, the quantum Hamiltonian of the model is

$$\hat{H} = -\frac{1}{2}\sum_{i=}^{N} \frac{\partial^2}{\partial x_i^2} + \alpha(1+\alpha)\sum_{i<j} \frac{1}{(x_i - x_j)^2} + \frac{1}{2}\omega^2 \sum_{i=1}^{N} x_i^2, \tag{14}$$

where x_i represents the position of the ith particle, ω represents the frequency of the external harmonic well and $\alpha \in [-1, 0]$ represents the coupling strength of mutually attractive interaction between the particles. In addition, the many-body wavefunction must vanish at $x_i = x_j$ for any pair $(i \neq j)$ of coordinates for $\alpha \neq 0$. It turns out that while $\alpha = 0$ represents noninteracting bosons (where the many-body wavefunction is symmetric under the exchange of x_i and x_j), $\alpha = -1$ represents noninteracting fermions (where the wavefunction vanishes at $x_i = x_j$). For other values of α, this model is known to exhibit fractional statistics (see, in particular, in the next subsection its manifestation in the thermodynamics of the model).

The many-body energy spectrum of this model is known exactly [9]. The energy $E(\{h_j\})$ is labeled by nonincreasing integers $h_1 \geqslant h_2 \geqslant h_3 \cdots \geqslant 1$:

$$E(\{h_j\}) = \omega \left[\sum_{j=1}^{N} h_j - \frac{1}{2}\alpha N(N-1) \right]. \tag{15}$$

By making a shift as in equation (8), i.e., defining a new set of variables $h'_j = h_j + \alpha(N - j)$, one can express the energy as $E = \omega \sum_{j=1}^{N} h'_j$ with the constraint that $(h'_j - h'_{j+1}) \geqslant -\alpha$. Thus, the spectrum of the Calogero model in a harmonic potential corresponds exactly to partition configurations of the MDP with parameter $p = -\alpha$, but now p is a real number such that $0 \leqslant p \leqslant 1$. Hence, the micro-canonical partition function of the Calogero model $\rho_{\mathrm{Cal}}(E, N)$ denoting the number of configurations with energy E and number of particles N is directly related to the number of partitions $\rho_p(E, N)$ of the MDP model via

$$\rho_{\mathrm{Cal}}(E, N) = \rho_p(E/\omega, N). \tag{16}$$

This implies that the grand-canonical partition functions of the two models are also related. Let $Z_{\mathrm{Cal}}(\beta, z) = \sum_{E,N} \rho_{\mathrm{Cal}}(E, N)\, \mathrm{e}^{-\beta E} z^N$ be the grand-canonical partition function in the Calogero model in a harmonic well of frequency ω, where β is the inverse temperature and z is the fugacity. Similarly, we define $Z_p(\beta, z) = \sum_{E,N} \rho_p(E, N)\, \mathrm{e}^{-\beta E} z^N$ as the double generating function in the MDP problem with parameter p. The relation in equation (16) then translates into the following relation between the grand partition functions:

$$Z_{\mathrm{Cal}}(\beta, z) = Z_p(\omega\beta, z). \tag{17}$$

3.2. Thermodynamic equivalence to exclusion statistics

Exclusion statistics can be most conveniently defined in the following thermodynamical sense. Let $Z(\beta, z)$ denote the grand partition function of a quantum gas of particles at inverse temperature β and fugacity z. Such a gas is said to obey exclusion statistics with parameter $0 \leqslant p \leqslant 1$ if $Z(\beta, z)$ can be expressed as an integral representation

$$\ln Z(\beta, z) = \int_0^\infty \tilde{\rho}(\epsilon) \ln y_p(z\,\mathrm{e}^{-\beta\epsilon})\,\mathrm{d}\epsilon, \tag{18}$$

where $\tilde{\rho}(\epsilon)$ denotes an effective single particle density of states and the function $y_p(x)$, which encodes fractional statistics, is given by the solution of the functional equation [5–7]

$$y_p(x) - x y_p^{1-p}(x) = 1. \tag{19}$$

Note that for $p = 0$, one gets $y_p(x) = 1/(1-x)$ and for $p = 1$, $y_p(x) = (1 + x)$. In these two extreme cases, equation (18) reduces to the standard grand partition functions of noninteracting bosons and fermions, respectively. The fractional statistics with parameter $0 < p < 1$ (that corresponds to an interacting gas) then smoothly interpolates between these two extreme cases.

There are at least two microscopic quantum models whose grand-canonical functions have the form of equation (18). The first example is the LLL anyon model [5] in the infinite volume limit which can be shown to satisfy equation (18) with an effective density of states $\tilde{\rho}(\epsilon) = \frac{BV}{\phi_0}\delta(\epsilon - \omega_c)$, where B is the external magnetic field, $\phi_0 = 2\pi/e$ is the flux quantum, $\omega_c = eB/2m$ is the cyclotron frequency and V is the infinite area of the system. In this model, the parameter $p = \phi/\phi_0$ corresponds to the flux carried by each anyon in units of the flux quantum. The second example corresponds to the one-dimensional Calogero model defined in equation (14) again in the infinite box limit. In this case, one can show that the grand partition function again can be written in the form as in equation (18) with an effective single particle density $\tilde{\rho}(\epsilon) = L/\sqrt{8\pi^2\epsilon}$ where L is the infinite length of the system. In both cases, the thermodynamics is computed in the presence of a long distance harmonic well regulator, and the thermodynamic limit where the external frequency $\omega \to 0$ is taken in such a way so that one correctly recovers the infinite box limit.

Here we show, using the equivalence to the MDP problem in equation (17), that the grand partition function of the one-dimensional Calogero model in an external harmonic well of frequency ω, in the limit $\omega \to 0$, can again be written in the general form as in equation (18), but now with an effective constant density of states $\tilde{\rho}(\epsilon) = 1/\omega$. Note that this is different from the Calogero model in an infinite box of size L (the second example mentioned in the preceding paragraph): here, the particles are sitting inside a harmonic well with almost vanishing but nonzero frequency.

To proceed, we first calculate the grand partition function of the MDP problem, $Z_p(\omega\beta, z) = \sum_{E,N} \rho_p(E, N)\,\mathrm{e}^{-\omega\beta E} z^N$, starting from equation (11). We set $x = \mathrm{e}^{-\beta\omega}$ in equation (11), multiply it by z^N and sum over N. Next we take the logarithm on both sides and then make a cluster expansion, $\ln Z_p(\omega\beta, z) = \sum_{n=1}^\infty b_n z^n$. Now, taking the $\beta\omega \to 0$ limit, one gets

$$b_1 \simeq \frac{1}{\omega\beta}\,\mathrm{e}^{-\omega\beta}, \qquad b_{n\geqslant 2} \simeq \frac{1}{\omega\beta}\,\frac{\mathrm{e}^{-n\omega\beta}}{n^2} \prod_{k=1}^{n-1}\left(1 - \frac{pn}{k}\right). \tag{20}$$

Note, on the other hand, that if one formally expands $\ln y_p(x)$ in equation (19) as a power series in x, one obtains

$$\ln y_p(x) = \sum_{n=1}^\infty \frac{x^n}{n} \prod_{k=1}^{n-1}\left(1 - \frac{pn}{k}\right). \tag{21}$$

Comparing equations (20) and (21) one infers that, provided the series is convergent, that is $z\,\mathrm{e}^{-\omega\beta} < 1$,

$$\ln Z_p(\omega\beta, z) = \int_1^\infty \ln y_p(z\,\mathrm{e}^{-\omega\beta\epsilon})\,\mathrm{d}\epsilon. \tag{22}$$

Making a further change of variable $\omega\epsilon \to \epsilon$, it follows that in the limit $\omega \to 0$ (keeping β fixed)

$$\ln Z_p(\omega\beta, z) \to \frac{1}{\omega}\int_0^\infty \ln y_p(z\,\mathrm{e}^{-\beta\epsilon})\,\mathrm{d}\epsilon. \tag{23}$$

This is again of the form in equation (18) with $\tilde{\rho}(\epsilon) = 1/\omega$. Using the equivalence in equation (17), we then conclude that the Calogero model in an external harmonic well with vanishing frequency, which precisely corresponds to the MDP problem with parameter $0 \leqslant p \leqslant 1$, can be viewed as a gas of particles obeying exclusion statistics with a statistical parameter $\alpha = -p$ and a constant density of states.

4. Partition asymptotics in MDP with $p \geqslant 0$

In this section, we explicitly compute the asymptotics of the probability distribution $P_p(N|E)$ in the MDP problem for all $p \geqslant 0$. We show that while the limiting shape of this distribution (properly centered and scaled) is Gumbel for $p = 0$, it is Gaussian for all $p > 0$ including the Fermi case $p = 1$.

4.1. Bosonic case $p = 0$

Our starting point is the generating function for $C_0(E, N)$ in equation (12). We formally invert this generating function using Cauchy's theorem and write

$$\begin{aligned} C_0(E, N) &= \frac{1}{2\pi\mathrm{i}}\int \frac{\mathrm{d}x}{x^{E+1}}\frac{1}{(1-x)(1-x^2)\cdots(1-x^N)} \\ &= \frac{1}{2\pi\mathrm{i}}\int \mathrm{d}\beta \exp\left[\beta E - \sum_{k=1}^{N}\ln(1-\mathrm{e}^{-\beta k})\right], \end{aligned} \tag{24}$$

where the integration is in the complex x plane along a contour around the origin and we have made a change of variable $x = \exp(-\beta)$ in going to the second line. For large E, one can then analyze the leading asymptotic behavior by employing the saddle-point method in the complex β plane. Anticipating that for large E, the most important contribution to the integral will come from small β, we first obtain the leading small β behavior of the action $S_{E,N}(\beta) = \beta E - \sum_{k=1}^{N}\ln(1-\mathrm{e}^{-\beta k})$. Using the Euler–Mclaurin summation formula, one can easily show that in the limit of $\beta \to 0$, $N \to \infty$ limit but keeping βN fixed the action can be written as

$$S_{E,N}(\beta) \simeq \beta E + \frac{1}{\beta}\int_0^{\beta N}\frac{t\,\mathrm{d}t}{\mathrm{e}^t - 1} - N\ln(1-\mathrm{e}^{-\beta N}). \tag{25}$$

We next maximize the action with respect to β, i.e., we set $\partial S/\partial\beta = 0$ to get

$$E = \frac{1}{\beta^2}\int_0^{\beta N}\frac{t\,\mathrm{d}t}{\mathrm{e}^t - 1}. \tag{26}$$

For a given large E, one gets β^* by implicitly solving the saddle-point equation (26) and substitute it back into the action $S_{E,N}(\beta^*)$. Thus, to leading order,

$$C_0(E, N) \simeq \exp[S_{E,N}(\beta^*)] \tag{27}$$

where the micro-canonical entropy $S_{E,N}(\beta^*)$ can be written as

$$S_{E,N}(\beta^*) = \frac{1}{\beta^*}\left[2\int_0^{\beta^* N} \frac{t\,\mathrm{d}t}{\mathrm{e}^t - 1} - \beta^* N \ln(1-\mathrm{e}^{-\beta^* N})\right]. \tag{28}$$

To bring out the scaling form of $C_0(E, N)$ explicitly for large E and N, we next proceed as follows. It is evident from the structure of the saddle-point solution that $\beta^* \sim E^{-1/2}$ for large E, whereas $\beta^* \sim 1/N$ for large N indicating that the correct scaling variable is $x = N/\sqrt{E}$. Next we set $\beta^* N = H(x)$. In terms of these new scaling variables, the saddle-point solution in equation (26) can be recast as

$$\frac{H^2(x)}{x^2} = \int_0^{H(x)} \frac{t\,\mathrm{d}t}{\mathrm{e}^t - 1}. \tag{29}$$

Thus, given x, one has to find $H(x)$ by implicitly solving equation (29). The entropy in equation (28) becomes $S_{E,N}(\beta^*) = \sqrt{E}g(x)$ where the scaling function $g(x)$ is given from equation (29) as

$$g(x) = 2\frac{H(x)}{x} - x\ln(1-\mathrm{e}^{-H(x)}). \tag{30}$$

Thus, asymptotically for large N and E, keeping the ratio $x = N/\sqrt{E}$ fixed, the cumulative number of configurations $C_0(E, N)$ for bosons can be written as

$$C_0(E, N) \simeq \exp\left[\sqrt{E}g\left(\frac{N}{\sqrt{E}}\right)\right], \tag{31}$$

where $g(x)$ is the large deviation function given exactly by equations (30) and (29). This is the main result of this subsection.

The function $g(x)$ has to be determined numerically by solving the implicit equations (30) and (29). A plot of this function is given in figure 3. The asymptotic properties of $g(x)$ for small and large x can be worked out easily. It can be shown that

$$\begin{aligned} g(x) &\approx -2x\ln(x) &&\text{as} \quad x \to 0 \\ &\approx a - \frac{2}{a}\exp(-ax/2) &&\text{as} \quad x \to \infty \end{aligned} \tag{32}$$

where $a = \pi\sqrt{2/3} = 2.5651\ldots$.

The result $g(x \to \infty) = a$ implies, from equation (31), that $\rho(E) = C(E, N \to \infty) \sim \exp[a\sqrt{E}]$ to leading order for large E, thus recovering the famous Hardy–Ramanujan result [1]. The normalized cumulative distribution of N (given E), $Q_0(N|E) = C_0(E, N)/\rho(E)$, then has the large deviation form

$$Q_0(N|E) \simeq \exp\left[-\sqrt{E}\Phi\left(\frac{N}{\sqrt{E}}\right)\right], \qquad \text{where} \quad \Phi(x) = a - g(x) \tag{33}$$

and $\Phi(x)$ has the asymptotic behavior

$$\begin{aligned} \Phi(x) &\approx a + 2x\ln(x), &&\text{as} \quad x \to 0 \\ &\approx \frac{2}{a}\exp(-ax/2), &&\text{as} \quad x \to \infty. \end{aligned} \tag{34}$$

As $x \to \infty$, i.e., as $N \gg \sqrt{E}$, clearly $Q(N|E) \to 1$ as expected, since it is the normalized cumulative distribution of N. The precise approach to 1 can be obtained using the large x asymptotics of $\Phi(x)$ in equation (34). Substituting this behavior into equation (33) one gets for $N \gg \sqrt{E}$

$$Q_0(N|E) \simeq \exp\left[-\frac{2}{a}\sqrt{E}\exp(-aN/2\sqrt{E})\right] = F_0\left(\frac{a}{2\sqrt{E}}(N - N_0^*(E))\right), \tag{35}$$

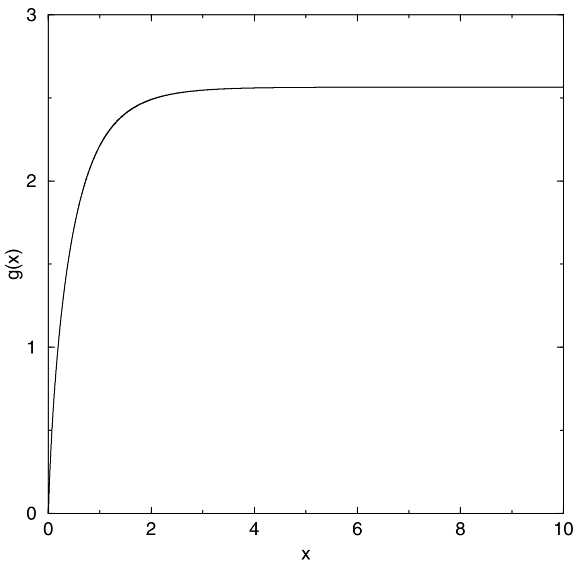

Figure 3. The large deviation function $g(x)$ for bosons ($p = 0$).

where the characteristic value of the random variable N is $N_0^*(E) \simeq \frac{1}{a}\sqrt{E}\log(4E/a^2)$ and the scaling function has the Gumbel form, $F_0(z) = \exp[-\exp[-z]]$. Evidently, the probability distribution $P_0(N|E) = Q(N|E) - Q(N-1|E) \simeq \partial Q_0(N|E)/\partial N$ has the scaling form

$$P_0(N|E) \simeq \frac{a}{2\sqrt{E}} F_0'\left(\frac{a}{2\sqrt{E}}(N - N_0^*(E))\right) \qquad \text{where} \quad F_0'(z) = \exp[-z - \exp[-z]] \tag{36}$$

which is highly asymmetric around the peak at $N = N_0^*(E)$. This limiting distribution of N that describes the probability of *typical* fluctuations of N of $\sim O(\sqrt{E})$ around the peak at $N_0^*(E)$ was originally derived by Erdös and Lehner by computing upper and lower bounds to the probability [16]. Our method allows us to obtain a more general result in equation (33) which is valid over a wider range and reduces to the Gumbel limiting form near the peak. A rigorous mathematical derivation of this result, including the exponential prefactor, can be found in the work of Szekeres [19].

4.2. The case $p > 0$

For $p > 0$, one can directly obtain the asymptotic behavior of $\rho_p(E, N)$ by using the identity in equation (13) and the already derived asymptotic behavior of $C_0(E, N)$ in equation (31). In the scaling limit when N and E are both large but the ratio $x = N/\sqrt{E}$ is kept fixed, one gets to leading order

$$\rho_p(E, N) \simeq \exp\left[\sqrt{E} f_p\left(\frac{N}{\sqrt{E}}\right)\right] \tag{37}$$

with

$$f_p(x) = \sqrt{1 - px^2/2}\, g\left(\frac{x}{\sqrt{1 - px^2/2}}\right) \tag{38}$$

where $g(x)$ is given in equations (30) and (29).

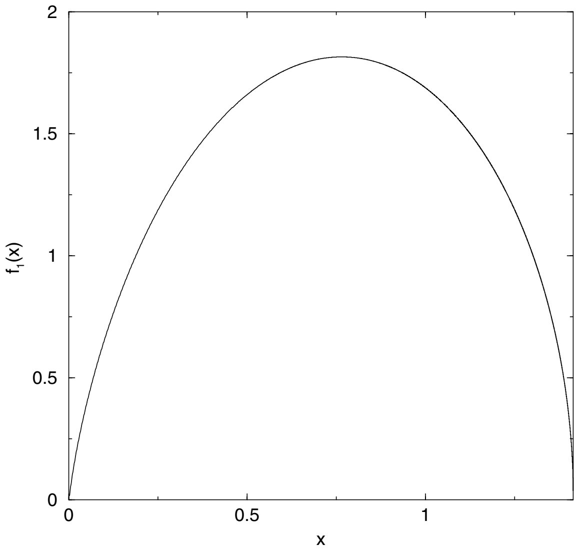

Figure 4. The large deviation function $f_1(x)$ for fermions ($p = 1$).

Note that the function $f_p(x)$ has nonzero support only over $x \in [0, \sqrt{2/p}]$. This is easy to understand from the fact that for $p > 0$, E has a minimum value for any given N or, equivalently, N has a finite maximum value for any given E. For example, for the fermionic case ($p = 1$), the lowest value of E for a given N corresponds to the Fermi energy $E_F = N(N+1)/2$ where one puts one fermion at each single particle level $\epsilon_i = i$ for $i = 1, 2, \ldots, N$. Thus, $E \geqslant N(N+1)/2$ for all N. In other words, for large N, $N \leqslant \sqrt{2E}$, i.e., $x \leqslant \sqrt{2}$. Similar arguments can be given for any positive $p > 0$. Unlike the function $g(x)$ which is monotonically increasing, the function $f_p(x)$ in equation (38) is a non-monotonic function in $x \in [0, \sqrt{2/p}]$. It vanishes at the two ends as

$$\begin{aligned} f_p(x) &\approx -2x\ln(x) \qquad \text{as} \quad x \to 0 \\ &\approx \frac{\pi\sqrt{6}}{3}(2p)^{1/4}\sqrt{\sqrt{2/p} - x} \end{aligned} \tag{39}$$

and has a unique maximum at $x^*(p) = a_1(p)$, where $a_1(p)$ can be obtained by setting $\mathrm{d}f_p(x)/\mathrm{d}x = 0$ in equation (38) and then using the known properties of $g(x)$. A plot of $f_p(x)$ for $p = 1$ (Fermi case) is given in figure 4.

By playing around with the form of $f_p(x)$ in equation (38) and that of $g(x)$ in equations (30) and (29) one can derive a number of explicit results. We skip the details here and just mention the results. For example, the location of the maximum $x^*(p) = a_1(p)$ is given by

$$a_1(p) = \frac{\ln y_0^\star}{\sqrt{p(\ln y_0^\star)^2/2 + Li_2(1 - 1/y_0^\star)}}, \tag{40}$$

where $y_0^\star - {y_0^\star}^{1-p} = 1$ and $Li_2(z) = \sum_{k=0}^{\infty} \frac{z^k}{k^2}$ is the dilogarithm function. For example, in the fermionic case $p = 1$, we get $y_0^\star = 2$ and $a_1(1) = 2\sqrt{3}\ln(2)/\pi = 0.764304\ldots$. Similarly, the value of the function at the maximum $f_p(x = a_1(p))$ can be shown to be

$$f_p(x = a_1(p)) = 2\sqrt{Li_2(1 - 1/y_0^\star) + p(\ln y_0^\star)^2/2}. \tag{41}$$

For example, for $p = 1$, it gives $f_p(x = a_1(1)) = \pi/\sqrt{3}$. For an arbitrary p, this formula which goes back to Meinardus [21] provides a generalization of the Hardy–Ramanujan formula for $\rho(E)$. One can check that equation (41) coincides with the integral formula obtained by Blencowe *et al* [20] who also investigated the Haldane statistics but did not provide any combinatorial interpretation of their result. Around the maximum at $x = a_1(p)$, the function $f_p(x)$ can be expanded in a Taylor series and up to the quadratic order

$$f_p(x) \simeq f_p(a_1(p)) + \frac{1}{2a_2^2(p)}(x - a_1(p))^2 + \cdots \tag{42}$$

where $a_2(p)$ can also be evaluated. For example, for $p = 1$, we get

$$a_2(1) = \left[\frac{3\pi^2 - 36\ln^2(2)}{\sqrt{3}\pi^3}\right]^{1/2} = 0.478815\ldots. \tag{43}$$

Evidently, one can easily evaluate the asymptotic behavior of $\rho_p(E) = \sum_N \rho_p(E, N)$ for large E by replacing the sum by an integral, use the large deviation form in equation (37) for $\rho_p(E, N)$ and then using the saddle-point method. To leading order, this gives

$$\rho_p(E) \simeq \exp[f_p(a_1(p))\sqrt{E}]. \tag{44}$$

The normalized probability distribution of N (for fixed E), $P_p(N|E) = \rho_p(E, N)/\rho_p(E)$, then has the large deviation asymptotics:

$$P_p(N|E) \simeq \exp\left[-\sqrt{E}\psi_p\left(\frac{N}{\sqrt{E}}\right)\right], \qquad \text{where} \quad \psi_p(x) = f_p(a_1(p)) - f_p(x). \tag{45}$$

Thus, for all $p > 0$, the probability distribution $P_p(N|E)$ has a peak at a characteristic value $N_p^*(E) = a_1(p)\sqrt{E}$ (note the difference from the boson case $p = 0$, where $N_0^*(E) \sim \sqrt{E}\ln(E)$). Using the expansion in equation (42), it follows that in the vicinity of $N_p^*(E)$ (over a scale of $\sim O(E^{1/4})$) $P_p(N|E)$ has a Gaussian limiting form

$$P_p(N|E) \simeq \frac{1}{a_2(p)E^{1/4}}F'\left(\frac{(N - a_1(p)\sqrt{E})}{a_2(p)E^{1/4}}\right), \qquad \text{where} \quad F'(z) = \frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-z^2/2}. \tag{46}$$

Note, in particular, that the standard deviation measuring the root-mean-squared fluctuation of N, $\sigma_p(E) = \sqrt{\langle(N - N_p^*(E))^2\rangle}$, grows with E as a power law, $\sigma_p(E) \simeq a_2(p)E^{1/4}$, where the exponent $1/4$ is universal for all $p > 0$. Moreover, apart from nonuniversal p-dependent scale factors such as $a_1(p)$ and $a_2(p)$, the full distribution $P_p(N|E)$ also has the same universal Gaussian limiting form for all $p > 0$. Thus, the Fermi point $p = 1$ is a generic point that is representative of all values of $p > 0$ as far as the limiting distribution is concerned. In this sense, all $p > 0$ behavior is controlled by the attractive Fermi fixed point as shown in figure 1. The bosonic fixed point at $p = 0$, on the other hand, is a repulsive one.

5. Summary and open problems

To summarize, in this paper we have provided a combinatorial interpretation of exclusion statistics in terms of minimal difference partitions. This correspondence is based on the observation that the grand-canonical partition function of the Calogero model coincides with the generating function of MDP. By going to the grand-canonical ensemble and taking a suitable thermodynamic limit, we have recovered the functional equation characteristic of exclusion statistics. Apart from establishing this correspondence, we have also provided a detailed analysis of the asymptotic behavior of $\rho_p(E, N)$. Our approach uses a mapping with the bosonic problem which holds for arbitrary $p \in [0, 1]$. In physical terms, this generalizes the well-known mapping between fermions and bosons with a linear dispersion law [22, 23]. The fact that this mapping has a number theoretical interpretation was apparently not known before. By using this mapping, we obtain a general description of the limiting laws of $P_p(N|E)$ for all $p > 0$. We find that the bosonic point is a repulsive fixed point where the statistics is Gumbel. In contrast, for all $p > 0$ the distribution is Gaussian. Several questions emerge from this work and would be worth investigating.

(1) The regime $p < 0$. In this case, the functional equation (19) still holds where $y_p(x)$ can be interpreted as the generating function of connected clusters on a p-ary tree. A preliminary investigation of this model shows that the scaling behavior of $\rho_p(E, N)$ is quite different from the previous case [24].

(2) In this work, we have limited ourselves to the integer partition problem or equivalently to a quantum gas of particles with equidistant single particle spectrum, i.e., with a constant density of states $\tilde{\rho}(\epsilon) = \text{constant}$. It would be interesting to investigate general partitions of the form $E = \sum n_i i^s$ that corresponds to having a power-law density of states, $\tilde{\rho}(\epsilon) \sim \epsilon^{1/s-1}$. In this case, we have shown in a recent work [25] that the bosonic sector gives rise to the three universal distribution laws of extreme statistics, namely the Gumbel, Weibull and Fréchet distributions. It would be interesting to explore the general $p > 0$ case including the fermionic sector and see if the bosonic point $p = 0$ is still a repulsive fixed point.

(3) For the bosonic case ($p = 0$), Vershik [26] and Temperley [27] calculated the limiting shapes of the Young diagram, i.e., the average height profile for a fixed but large E. This result can be generalized [24] to the case $p > 0$ using the functional equation (19).

Note added in proof. After completion of this work, we came to know of [28] where the author also used equation (38) to derive the asymptotics of minimal difference partitions. We thank Dan Romik for pointing out this reference to us.

Appendix. A grand-canonical approach

Let us denote by $\rho_{p,s}(E, N)$ the number of ways to partition a positive integer E into N non-negative integer parts, $E = \sum_{j=1}^{N} h_j$, such that each summand exceeds the next by at least an integer p, i.e., $(h_j - h_{j+1}) \geqslant p$ for all $j = 1, 2, \ldots, N-1$ and such as the smallest part is an integer $\geqslant s$. This MDP generalizes the standard MDP partitioning case studied above where the smallest part s was set to 1, i.e., $\rho_p(E, N) = \rho_{p,1}(E, N)$. The canonical partition function in (11) now becomes

$$\sum_{E=1}^{\infty} \rho_{p,s}(E, N) x^E = \frac{x^{sN+pN(N-1)/2}}{(1-x)(1-x^2)\cdots(1-x^N)}. \tag{A.1}$$

The generating function for $\rho_{p,l}(E, N)$ is $Z_{p,l}(x, z) = \sum_{E,N}^{\infty} \rho_{p,l}(E, N) x^E z^N$, where by convention the zeroth-order term in z is equal to 1. Setting $x = \mathrm{e}^{-\beta}$, analogous considerations as in section 3 give in the small β limit

$$\ln Z_{p,s}(\beta, z) = \int_s^{\infty} \ln y_p(z\,\mathrm{e}^{-\beta\epsilon})\,\mathrm{d}\epsilon \tag{A.2}$$

which generalizes (22) to the case $s \neq 1$ (here the parameter ω in (22) is set to 1). This is again exclusion statistics thermodynamics, with a constant one-body density of state starting at $\epsilon = s$. It means that the MDP problem can be equivalently viewed, in the small β limit, as a gas of particles obeying exclusion statistics with a statistical parameter $\alpha = -p$ and a constant density of states.

In the thermodynamic limit, one is interested by the large E and large N behavior of $\rho_{p,s}(E, N)$. In the grand-canonical ensemble, this amounts to evaluating the thermally averaged partitioned integer $\langle E\rangle \equiv -\frac{\partial \ln Z_{p,s}(\beta,z)}{\partial\beta}$ and the average number of parts $\langle N\rangle \equiv z\frac{\partial \ln Z_{p,s}(\beta,z)}{\partial z}$. Both $\langle E\rangle$ and $\langle N\rangle$ are given in terms of n_p, the mean occupation number at 'part' ϵ, namely $\langle E\rangle = \int_s^\infty n_p(z\,\mathrm{e}^{-\beta\epsilon})\epsilon\,\mathrm{d}\epsilon$ and $\langle N\rangle = \int_s^\infty n_p(z\,\mathrm{e}^{-\beta\epsilon})\,\mathrm{d}\epsilon$. It satisfies $y_p = 1 + \frac{n_p}{1-pn_p}$ such that

$$z\,\mathrm{e}^{-\beta\epsilon} = \frac{n_p}{(1+(1-p)n_p)^{1-p}(1-pn_p)^p} \tag{A.3}$$

implying in particular that $n_p \leqslant 1/p$, i.e., the mean occupation number at part ϵ cannot exceed $1/p$.

Pushing further this grand-canonical analysis, it is easy to obtain that

$$\langle E\rangle - s\langle N\rangle = \frac{1}{\beta}\ln Z_{p,s}(\beta, z) \qquad \text{and} \qquad \langle N\rangle = \frac{1}{\beta}\ln y_p(z\,\mathrm{e}^{-\beta s}). \tag{A.4}$$

The entropy $S \equiv \ln Z_{p,s}(\beta, z) + \beta\langle E\rangle - (\ln z)\langle N\rangle$ rewrites

$$S = 2\beta\left(\langle E\rangle - s\langle N\rangle - \frac{p}{2}\langle N\rangle^2\right) - \langle N\rangle\ln(1-\mathrm{e}^{-\beta\langle N\rangle}), \tag{A.5}$$

with

$$\beta^2(\langle E\rangle - s\langle N\rangle - \frac{p}{2}\langle N\rangle^2) = -\int_0^{1-\mathrm{e}^{-\beta\langle N\rangle}} \frac{\ln(1-u)}{u}\,\mathrm{d}u = \int_0^{\beta\langle N\rangle} \frac{u}{\mathrm{e}^u - 1}\,\mathrm{d}u. \tag{A.6}$$

Equations (A.5) and (A.6) are the building blocks from which the entropy can be obtained: inverting (A.6) gives β as a function of $\langle E\rangle$ and $\langle N\rangle$ so that the entropy in (A.5) becomes a function of $\langle E\rangle$ and $\langle N\rangle$ only, and consequently $\rho_{p,s}(\langle E\rangle, \langle N\rangle) \simeq \exp(S)$ as well. Note finally that (A.5) and (A.6) are nothing but the generalization of (29) and (30) to the MDP $p \neq 0$ and $s \neq 0$ case. They are also, at the canonical level, contained in (13) (in the case $s = 1$).

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